

Non-classicality and quandle difference invariants

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Abstract

Non-classical virtual knots may have upper and lower quandles which are not isomorphic. We exploit this property to define quandle difference invariants, which can detect non-classicality by comparing the numbers of homomorphisms into a finite quandle from a virtual knot's upper and lower quandles. The invariants for small-order finite quandles detect non-classicality in several interesting virtual knots. We compute the difference invariant with the six smallest connected quandles for all non-evenly intersticed Gauss codes with 4 crossings. For non-evenly intersticed Gauss codes with 4 crossings, the difference invariants detect non-classicality in 86% of codes which have non-trivial upper or lower counting invariant values.

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1 Introduction

Virtual knots are equivalence classes of Gauss codes under the equivalence relation determined by Reidemeister moves. A Gauss code is *realizable* if it determines a planar knot diagram; virtual knots which include realizable Gauss codes are *classical* and those which do not are *non-classical*. See [12] and [2] for more.

Detecting the non-classicality of a virtual knot is not always simple since each classical knot type includes some Gauss codes which cannot be realized without virtual crossings. Various methods have been proposed for detecting non-classicality of Gauss codes; see [13] and [5].

In this paper we define *quandle difference invariants* which can detect non-classicality of certain virtual knots. We show that quandle difference invariants detect non-classicality in some interesting examples of virtual knots, including one virtual knot whose non-classicality is not detected by the (upper) knot quandle or by the one or two strand bracket polynomials, though its non-classicality is detected with the three-strand bracket polynomial in [5]. We describe an algorithm and provide an implementation in Maple for computing these invariants from a Gauss code and a finite quandle matrix.

In order to test the effectiveness of quandle difference invariants at detecting non-classicality, we generated a list of all Gauss codes with three and four crossings, eliminating codes which are obviously classical or trivial. From this list, we then computed the quandle difference invariant for each such code; the results are collected in table 1.

The paper is organized as follows. Section 2 gives a brief review of virtual knots, section 3 introduces the quandle difference invariants and 2-component invariants, and section 4 contains computational results. The `Maple` code used to obtain these results is available at www.esotericka.org/quandles; bugfixes and improvements will be made as necessary.

2 Virtual knots and non-classicality

We begin with a definition from [12].

Definition 1 Let K be an oriented knot diagram with crossings labeled with names and signs. A *Gauss code* is a sequence of crossing labels with over/under and sign information recorded in the order encountered while following the orientation of the knot; it is defined up to cyclic permutation. A *Gauss diagram* is obtained from a Gauss code by writing the code counterclockwise around a circle and joining the two instances of each crossing with an arrow oriented “in the direction of gravity”, that is, toward the undercrossing label, and noting the sign for each such arrow.

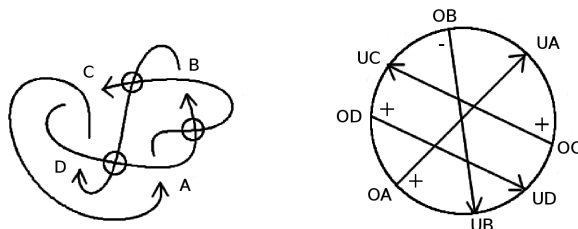
A Gauss code or diagram determines a knot diagram in a neighborhood of each crossing and specifies which strands are to be connected; thus a knot diagram can be reconstructed from its Gauss code. We can then consider doing knot theory combinatorially by defining a “knot” to be an equivalence class of Gauss codes under the equivalence relation determined by the Reidemeister moves. However, there is no guarantee that a given Gauss code determines a planar knot diagram; we may be forced to introduce additional crossings not mentioned in the code in order to draw the diagram in the plane. These *virtual crossings* are drawn as circled self-intersections, and since they’re not really there, we’re permitted to redraw any section of an arc with only virtual crossings on its interior arbitrarily provided we put only virtual crossings in the interior.

Definition 2 A *virtual knot* is an equivalence class of Gauss codes under the equivalence relation generated by the Reidemeister moves.

Example 1 The Gauss code

$$UA + OB - UC + OD + OA + UB - UD + OC +$$

corresponds to the pictured virtual knot diagram and Gauss diagram.



Virtual crossings may be avoided by drawing our non-planar knot diagrams on a surface with genus; we may then regard the Gauss code as describing a knot in a thickened surface $\Sigma \times [0, 1]$. The equivalence relation defining virtual knots then translates into isotopy of the knot in $\Sigma \times [0, 1]$ modulo stabilization and destabilization moves [2], [14].

Classical knot theory forms a subset of virtual knot theory, since every classical knot may be considered a virtual knot without virtual crossings, and, crucially, virtually isotopic classical knots are classically isotopic [8], [14], [16]. Many invariants of classical knots extend to virtual knots by simply ignoring virtual crossings; these include the knot group, the knot quandle and biquandle, the various skein polynomials, finite type invariants, etc.

Given a Gauss code, one asks whether the Gauss code corresponds to a classical knot. If reconstruction yields a planar knot diagram, then the answer is clearly yes; however, every classical knot has representative Gauss codes which determine non-planar, i.e., virtual, diagrams, [16]. Thus, if a Gauss code is not obviously classical, the code may well represent either a classical or non-classical virtual knot. In the next section, we discuss a method for detecting non-classicality of virtual knots using quandle difference invariants.

3 Quandle difference invariants

Definition 3 A *quandle* is a set Q with an operation $\triangleright : Q \times Q \rightarrow Q$ satisfying

- (i) for all $a \in Q$, $a \triangleright a = a$,
- (ii) for every $a, b \in Q$ there is a unique $c \in Q$ such that $a = c \triangleright b$, and
- (iii) for all $a, b, c \in Q$, $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

The term *quandle* is due to Joyce; in [11], a quandle is associated to each knot diagram by a Wirtinger-type presentation. By construction, this knot quandle is an invariant of Reidemeister moves and hence virtual isotopy. While it is difficult to directly compare quandles given by presentations, we can exploit the fact that knot quandles are finitely presented to obtain a computable invariant of quandle isomorphism type by counting homomorphisms onto a finite quandle. This quandle counting invariant is studied in various papers and has been jazzed up with quandle 2-cocycles from various quandle homology theories [6], [4], [1].

In [8] it is observed that virtual knots have two knot groups, the *upper group* obtained by the usual Wirtinger presentation and the *lower group* obtained by first flipping the virtual knot over, i.e., looking at the knot diagram from the other side of its supporting surface, and then using the usual Wirtinger presentation. For classical knots, flipping over is an ambient isotopy, so for classical knots, the upper and lower groups are isomorphic. However, flipping over virtual knots involves doing forbidden moves which potentially change the virtual knot type (see [17]), so in general a virtual knot can have non-isomorphic upper and lower groups. Exactly the same observation applies to quandles [12].

Theorem 1 *Let K be a virtual knot diagram and let $\text{Flip}(K)$ be the virtual knot diagram obtained from K by viewing K from a base point on the other side of the supporting surface. Then $\text{Flip}(K)$ is an invariant of virtual isotopy.*

Proof. Reidemeister and virtual Reidemeister moves on the flipped knot are simply Reidemeister and virtual Reidemeister moves on the original knot. \square

Corollary 2 *Let Φ be any virtual knot invariant. Then*

$$\Phi_2(K) = (\Phi(K), \Phi(\text{Flip}(K)))$$

is an invariant of virtual isotopy.

The two-component version of a virtual knot invariant does not reveal any additional information about K when K is classical, but they can be very helpful in distinguishing non-classical virtual knots. Obvious examples of 2-component invariants include the 2-component counting invariants from the knot group, quandle and biquandle; 2-component CJKLS 2-cocycle invariants; the 2-component Jones polynomial, etc.

Corollary 2 suggests the following

Definition 4 Let T be a finite quandle and K a virtual knot. Let U denote the upper quandle of K and L the lower quandle of K . The *two-component counting invariant* is the ordered pair

$$Q2_T(K) = (|\text{Hom}(U, T)|, |\text{Hom}(L, T)|).$$

The *quandle difference invariant* $QD_T(K)$ of K associated to T is the difference between the number of the number of homomorphisms from U to T and the number of homomorphisms from L to T . That is,

$$QD_T(K) = |\text{Hom}(U, T)| - |\text{Hom}(L, T)|.$$

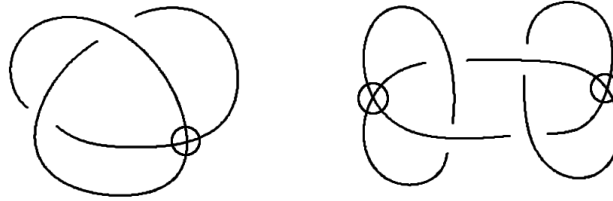
The fact that classical knots have isomorphic upper and lower quandles implies

Theorem 3 *If a virtual knot K is classical, then $QD_T(K) = 0$ for every finite quandle T .*

Corollary 4 *If $QD_T(K) \neq 0$ for any finite quandle T , then K is not classical. In particular, if $QD_T(K) \neq 0$ for any finite quandle T , then K is not the unknot.*

Moreover, since the individual components of QD_T are invariants of virtual isotopy, QD_T is itself an invariant of virtual knots; $QD_T(K) \neq QD_T(K')$ implies K is not virtually isotopic to K' . In the present work, however, we focus on using the quandle difference invariant to detect non-classicality of virtual knots. We know already that the quandle difference invariant does not always detect non-classicality.

Example 2



The knot on the left is Kauffman's virtual trefoil. It is known to be non-trivial and non-classical [12]. However, its upper and lower quandles are both isomorphic to the free quandle on one generator, that is, the knot quandle of the unknot. The knot on the right is a variant of the Kishino knot; it has rotational symmetry about the horizontal axis, so its upper and lower quandles have identical presentations.

Definition 5 We say a virtual knot is *pseudoclassical* if it is virtually isotopic to its flip. In particular, a pseudoclassical virtual knot has isomorphic upper and lower quandles.

Clearly, the quandle difference invariant cannot detect the non-classicality of pseudoclassical virtual knots for any finite quandle T . However, the quandle difference invariants are effective at detecting non-classicality in many virtual knots. Using the smallest connected quandle, for example, we detect the non-classicality (and hence non-triviality) of the non-alternating Kishino knot in figure 1. It only took two tries to find a quandle difference invariant which detects the non-classicality of the virtual knot K_D in 1; this knot is not distinguished from the unknot by either the Jones polynomial or 2-strand bracket polynomial, though the 3-strand bracket polynomial does the trick [5].

$$\begin{aligned}
 K &= \text{[Diagram of virtual knot K: two circles connected by two horizontal lines. The left circle has a crossing with a negative sign (arrowhead over arrowtail). The right circle has a crossing with a positive sign (arrowhead under arrowtail).]} \quad T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \\
 QD_T(K) &= -6 \\
 K_D &= \text{[Diagram of virtual knot K_D: two circles connected by two horizontal lines. The left circle has a crossing with a negative sign and an additional loop. The right circle has a crossing with a positive sign and an additional loop.]} \quad T_2 = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix} \\
 QD_{T_2}(K_D) &= -12
 \end{aligned}$$

Figure 1: QD_T detects non-classicality of virtual knots K and K_D .

To compute the upper and lower quandle presentations from a Gauss diagram, we divide the outer circle at each arrowhead to obtain a list of arcs in the diagram, and the quandle relations at each crossing are determined according to the crossing sign. As figure 2 shows, the rule is that if the arrowhead dividing x_i from x_{i+1} has tail on arc y , the relation determined is $x_i \triangleright y = x_{i+1}$ if the crossing sign is $+$ and $x_{i+1} \triangleright y = x_i$ if the crossing sign is $-$.

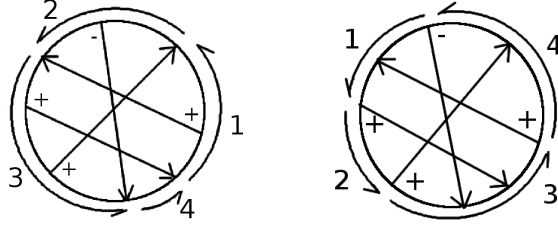
Dividing the same Gauss diagram at the arrowtails and applying the same procedure yields a presentation of the lower quandle L . Alternatively, we can obtain a presentation of the lower quandle by reversing the direction of the arrows, then getting the upper quandle presentation of the resulting diagram.

Example 3 Let K be the virtual knot from example 1. Then K has upper quandle presentation

$$U = \langle 1, 2, 3, 4 \mid 1 \triangleright 3 = 2, 2 \triangleright 1 = 3, 4 \triangleright 2 = 3, 4 \triangleright 3 = 1 \rangle$$

and lower quandle presentation

$$L = \langle 1, 2, 3, 4 \mid 1 \triangleright 3 = 2, 2 \triangleright 4 = 3, 3 \triangleright 1 = 4, 1 \triangleright 3 = 4 \rangle.$$



From these presentations, we can compute the quandle difference invariant $QD_T(K)$ for any choice of finite target quandle T by systematically considering all assignments of element of T to each generator in the quandle presentation and checking whether the assignment satisfies the required relations in the target quandle. In the next section, we describe our computational procedure for computing the quandle difference invariant for any choice of Gauss code and finite target quandle and give some computational results.

4 Computational results

Maple code implementing the algorithms below is available in the file `quandledifference.txt` at the second listed author's website www.esotericka.org/quandles.

We compute the quandle difference invariant $QD_T(K)$ using the quandle matrix notation from [9]. Specifically, let $T = \{x_1 \dots, x_n\}$ be a finite quandle. Then the matrix of T , M_T is the matrix abstracted from the operation table of T by dropping the xs , keeping only the subscripts. A list of all finite quandles with up to 5 elements appears in [9]; lists of all finite quandles with 6, 7 and 8 elements

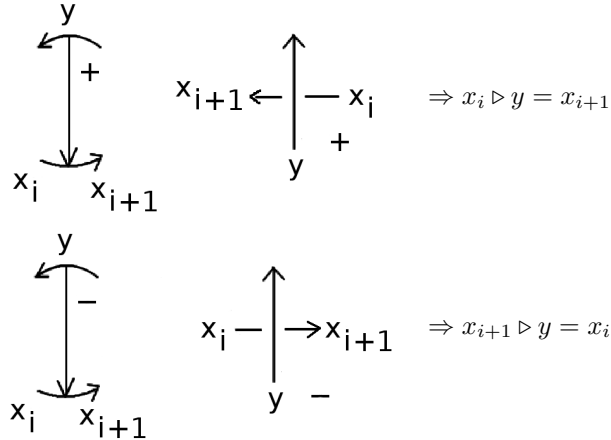


Figure 2: Quandle relations from a Gauss diagram.

along with software for computing such lists are available at www.esotericka.org/quandles.¹ The file `quandles-maple.txt` contains `Maple` code for counting homomorphisms from a knot quandle to finite quandle specified by a matrix.

We represent a finitely presented quandle with a *quandle presentation matrix* as described in [10]. That is, we put

$$M_{ij} = \begin{cases} k, & x_i \triangleright x_j = x_k, \\ 0, & \text{else.} \end{cases}$$

If a Gauss code has a negative undercrossing label x_i such that the next undercrossing label x_{i+1} has sign $+$ and both overcrossings lie on the same arc x_j , then the quandle relations determined are $x_i \triangleright x_j = x_{i-1}$ and $x_i \triangleright x_j = x_{i+1}$, which determine conflicting entries for the i, j position in the presentation matrix. In this situation we can do a type I Reidemeister move to obtain a presentation with no conflicting entries; the program `gfix` finds and corrects this problem.

Example 4 The virtual knot diagram in example 1 has upper and lower quandle presentation matrices

$$U = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We represent Gauss codes in `Maple` as vectors with (appropriately?) Gaussian integer entries, using the convention

$$\begin{aligned} OX+ &\leftrightarrow X, & UX+ &\leftrightarrow -X, \\ OX- &\leftrightarrow X+I, & UX- &\leftrightarrow -X-I, \quad X \in \mathbb{Z}. \end{aligned}$$

To allow for multi-component links, each component is ended with a “0”.

Example 5 The multicomponent Gauss code

$$U1 + O2 - O3 - U2-, \quad O1 + U3-$$

is represented in our `Maple` notation as the vector

$$[-1, 2 + I, 3 + I, -2 - I, 0, 1, -3 - I, 0].$$

Our quandle-difference computing algorithm then takes a Gauss code and first runs it through `gfix`, which detects the situation described above and does Reidemeister I moves as necessary until the code has an upper quandle presentation matrix with no conflicting entries. The program `gauss2pres` then reads off the upper quandle presentation matrix from the Gauss code. Our main program, `qdiff`, uses `homcount` from `quandles-maple.txt` to compute the number of homomorphisms from the upper quandle of the matrix returned by `gfix` into the specified target quandle. Then, the program simply multiplies each entry in the code by -1 to switch the direction of the arrows and cycles through to put the -1 or $-1 - i$ entry in the first position, then repeats the procedure to get the lower quandle presentation matrix and its counting invariant. Finally, the

¹An independently derived list of quandles with up to 6 elements, obtained by Yamada, is included in [3].

program reports the difference between these two numbers. We also include `q2chom` which computes the 2-component quandle counting invariant.

In order to get some feeling for the effectiveness of quandle difference invariants at detecting non-classicality of virtual knots, we generated lists of all non-evenly intersticed single-component Gauss codes with three and four crossings, removing codes which reduce by an obvious type I or type II move, using `rglist`. The non-evenly intersticed condition guarantees non-planarity of the corresponding virtual knot, which may be classical or non-classical (including pseudoclassical). There are 172 such 3-crossing Gauss codes; however, QD_T does not detect non-classicality in any of these 3-crossing codes for the six smallest connected quandles. There are 17040 such 4-crossing codes. We then computed the number of codes in which non-classicality was detected using some connected quandles of order up to six. The results are collected in table 1. It is not clear what percentage of the Gauss codes with $QD_T = 0$ are classical (i.e., diagrams of the unknot, trefoil, or figure eight), what percentage are pseudoclassical, and what percentage may have non-classicality detected by larger finite quandles, nor is it clear how many distinct virtual knots are represented. However, eliminating the codes which have trivial values for the quandle counting invariants for both upper and lower quandles for all six listed quandles, a step which eliminates the classical and pseudoclassical codes from the lists, gives a list of 16 3-crossing codes and 4140 4-crossing codes. Of these 4-crossing codes, 3570 or 86% have non-classicality detected by at least one of the six listed quandle difference invariants.

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Target quandle T	Number detected/ 17040 (4-crossing codes)
$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$	3060
$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$	1350
$\begin{bmatrix} 1 & 3 & 4 & 5 & 2 \\ 3 & 2 & 5 & 1 & 4 \\ 4 & 5 & 3 & 2 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$	492
$\begin{bmatrix} 1 & 4 & 5 & 3 & 2 \\ 3 & 2 & 4 & 5 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 1 & 2 & 4 & 3 \\ 4 & 3 & 1 & 2 & 5 \end{bmatrix}$	72
$\begin{bmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$	426
$\begin{bmatrix} 1 & 4 & 5 & 2 & 3 & 1 \\ 4 & 2 & 6 & 1 & 2 & 3 \\ 5 & 6 & 3 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 & 6 & 5 \\ 3 & 5 & 1 & 6 & 5 & 4 \\ 6 & 3 & 2 & 5 & 4 & 6 \end{bmatrix}$	3060

Table 1: Number of non-evenly intersticed single-component Gauss codes in which `qdiff` detects non-classicality.

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